

# Confidence Intervals

## QMET103



Library, Teaching and Learning



New Zealand's specialist land-based university

**CONFIDENCE INTERVALS** provide an interval estimate of the unknown population parameter.

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### **What is a confidence interval?**

Statisticians have a habit of hedging their bets. They always insert qualifiers into reports, warn about all sorts of assumptions, and never admit to anything more extreme than probable. There's a famous saying:

*"Statistics means never having to say you're certain."*

Statements must be qualified, of course, because we are always dealing with imperfect information. In particular, it is often necessary to make statements about a population using information from a sample. No matter how carefully this sample is selected to be a fair and unbiased representation of the population, relying on information from a sample will always lead to some level of uncertainty.

**So, a confidence interval is an *interval* within which we can estimate, with some *confidence*, that the true population parameter will lie.**

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### **Introduction**

Suppose we were interested in answering a simple research question such as:

"What is the mean number of digits that can be remembered?"

Having specified the population of people to be: "Lincoln University students", we take a sample of 10. The number of digits remembered for these 10 students is: 4, 4, 5, 5, 5, 6, 6, 7, 8, 9. From these results we find the estimated value of  $\mu$ , that is  $\bar{x}$ , to be 5.9 and  $s = 1.66$ .

But this will certainly not be a perfect estimate. It is bound to be at least either a little too high or a little too low.

For the estimate of  $\mu$  to be of value, we need to have some idea of how precise it is. That is, how close to  $\mu$  is the estimate likely to be?

An excellent way to specify the precision is to construct a confidence interval.

Since we know that approximately 68% of a distribution lies within 1 s.d. of the mean, we could say that we are 68% certain that the population mean lies within an interval of  $\bar{x} \pm 1s.d.$  That is, we could be about 68% confident that the true mean number of digits that can be remembered lies between  $5.9 \pm 1.66$  or between 4.24 and 7.56.

And, since we know that approximately 95% of a distribution lies within 2 s.d. of the mean, we could say that we are about 95% certain that the population mean lies within an interval of  $\bar{x} \pm 2s.d.$  or between  $5.9 \pm 2 \times 1.66$  i.e. between 0.92 and 10.88.

Similarly if approximately 99% of a distribution lies within 3 s.d. of the mean, we could say that we are about 99% certain that the population mean lies within an interval of  $\bar{x} \pm 3s.d.$  or between  $5.9 \pm 3 \times 1.66$  i.e. between 4.24 and 7.56.

### **Interpretation:**

A 95% confidence interval estimate means that if all possible samples are taken, 95% of them would include the true population mean somewhere in their interval.  
Or we can be 95% confident the interval contains the true population mean.  
(Other confidence intervals used more frequently are 90% CI or 99% CI).

## How is it calculated?

The formula for a confidence interval is

$$C.I. = \text{sample statistic} \pm Z_{\text{score}} \times \text{s.e. (population parameter)}$$

or  $C.I. = \text{sample statistic} \pm t_{\text{score}} \times \text{s.e. (sample statistic)}$

Each situation needs careful consideration, and the following decisions made:

- Is the sample statistic a mean or proportion?
- Is there one sample or two?
- What is the standard error (s.e.) of the sample statistic?
- Should a  $t$  or a  $Z_{\text{score}}$  be used?

A flow diagram may help to see the process. (see later)

**Notice:** The formula consists of **three** parts, separated by  $\pm$  and  $\times$ .  
In all C.I.'s, the expression after the  $\times$ -sign is the **standard error**.

## One sample, with mean and population standard deviation given ( $\sigma$ or $\sigma^2$ known)

That is, you are given a sample mean and the population standard deviation or variance.

$$\text{Use } C.I. = \bar{x} \pm Z_{\text{score}} \times \frac{\sigma}{\sqrt{n}}$$

1. From the sample, calculate  $\bar{x}$  or note it if is given.
2. Look up a  $Z_{\text{score}}$  from the standard table (\*see below). Note the level of confidence required.
3. Calculate the standard error of the sample statistic. For a mean, the s.e. is  $\frac{s}{\sqrt{n}}$ .

**Example:** For a set of data,  $\bar{x} = 85$ ,  $\sigma = 30$ , and  $n = 30$ , find a 95% C.I.

- $\bar{x} = 85$
- $Z_{\text{score}}$ :  $95\% = 0.95$ ;  $0.95 \div 2 = 0.475$ ;  $0.5 + 0.475 = 0.975$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0										
...										
1.9	0.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

That is,  $Z = 1.96$

- Hence 
$$C.I. = \bar{x} \pm Z_{\text{score}} \times \frac{\sigma}{\sqrt{n}} = 85 \pm 1.96 \times \frac{30}{\sqrt{30}}$$
$$= (74.26, 95.74)$$

### Interpretation:

"We can be 95% confident the true population mean lies between 74.26 and 96.74."

**Note on use of calculator:** The  $\pm$  in the formula means you must do two calculations. Use the "replay" key on your calculator for this. Calculate the lower value in the C.I., using the "minus" ( - ) key. Then, with the **right hand >** (which takes you to the beginning of the calculation) scroll across until the cursor is over the ( - ). Change to + and press =. You now have the upper value of the interval.

## One sample, with mean and sample standard deviation given

If the only information given is mean and *sample* standard deviation ( $\sigma$  or  $\sigma^2$  unknown) a  $t_{\text{score}}$  is used instead of a  $Z_{\text{score}}$

$$\text{Use } C.I. = \bar{x} \pm t_{(n-1)} \times \frac{s}{\sqrt{n}}$$

1. From the sample, calculate  $\bar{x}$ . (This may be given to you.)
2. Calculate the degrees of freedom. For a one sample mean, this is  $n - 1$ .
4. Look up a  $t_{\text{score}}$  from the  $t$ -table (\*see below). Note level of confidence required and use the correct degrees of freedom ( $df$ ).
5. Calculate the standard error of the sample statistic.

$$\text{For a one sample mean, } se(\text{mean}) = \frac{s}{\sqrt{n}}.$$

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**Example:** For a set of data,  $\bar{x} = 85$ ,  $s = 30$ , and  $n = 30$ , find a 95% C.I.

$se(\text{mean}) = \frac{30}{\sqrt{30}}$ , degrees of freedom  $(n - 1) = 29$  and 95%  $\Rightarrow$  upper tail = 0.025.

So from table:  $t_{\text{score}}$ :

df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	$t_{.001}$	$t_{.005}$
.....							
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
...							

That is,  $t_{.025} = 2.045$ .

$$\begin{aligned} \text{Hence, the C.I.} &= 85 \pm 2.045 \times \frac{30}{\sqrt{30}} \\ &= (73.7934, 96.2006) \end{aligned}$$

*That is, we can be 95% confident that the true population mean lies between approximately 73.79 and 96.20.*

Note that this interval is only slightly greater than the one calculated previously using population s.d and a  $Z_{\text{score}}$ .

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**Practise Questions – one sample**

1. A machine manufactures bolts to a set length with variance of 6.25 mm. A random sample of 20 bolts is checked and found to have a mean length of 75.2 mm. Find the 99% confidence interval for the mean length of the bolts.
2. 60 people were asked to measure their pulse rates after completing a 3 km run. The mean was 105 beats and the standard deviation was 8 beats. Construct a 95% confidence interval for the mean of the population of people.
3. A type of golf ball is tested, by dropping it onto a hard surface from a height of 1 metre. The height it bounces is known to be normally distributed with a standard deviation of 3.6 cm. If a sample of 100 balls are tested and the mean height of the bounces is 82 cm, find
  - a. 90%
  - b. 95%
  - c. 99%confidence intervals for the mean of the bounce of the golf ball.
4. A sample of stalactites (a type of rock formation) found in a glow worm cave produced the following lengths in cm:

9.6	16.9	15.1	14.3	15.9	17.2	13.0
17.1	15.4	16.2	4.5	20.3	21.2	15.7

Assuming that this sample came from a normal population, calculate a 95% confidence interval for the mean length of stalactites in the cave.
5. A doctor conducts a small survey with a random sample of his patients, measuring their cholesterol levels. Here is his data (the measurements are in m.mol/L):

3.6	6.9	5.1	4.2	5.5	7.2	3.0	5.8	4.9	9.9	7.1
5.4	6.2	4.5	6.3	8.2	5.7	4.4	7.9	3.2		

Find an 80% confidence interval for the mean cholesterol level of his patients.
6. A major department store chain is interested in estimation the average amount its credit card users spent on their first visit to the chain's new store. Fifteen credit cards were randomly sampled and analysed to show a mean of \$50.50 and variance 400. Construct a 95% confident interval for the average amount its credit card users spent on their first visit to the chain's new store assuming that the amount spent follows a normal distribution.
7. A race car driver tested his car for the time he takes to accelerate from 0 to 60 km/hr. In 20 such tests he obtained an average of 4.85 seconds with a standard deviation of 1.47 seconds. What is a 95% confidence interval for the acceleration time?
8. The actual voltages of power packs labelled as 12 volts are as follows: 11.77, 11.90, 11.64, 11.84, 12.13, 11.99, and 11.77. Calculate a 99% confidence interval for the true voltage in these packs.

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**When reading a question, note:**

- Has the variance, standard deviation or standard error been given?
  - Adjust your formula to match what has been given.
- Is the information from the population or the sample?
  - Remember to use a  $Z_{\text{score}}$  if it is from the population and  $t_{\text{score}}$  for a sample.

## One sample proportion given

$$\text{Use } C.I. = p \pm Z_{\text{score}} \times \sqrt{\left(\frac{p(1-p)}{n}\right)}$$

1. From the sample, calculate  $p$  or note if given.
2. Look up a  $Z_{\text{score}}$  from the standard table (\*see below). Note the level of confidence required.
3. Calculate the standard error of the sample statistic.

$$\text{For a proportion, the s.e. is } \sqrt{\left(\frac{p(1-p)}{n}\right)}.$$

### Example:

In a Rugby World Cup, a random sample of supporters were asked, "Which country do you think will win the 2003 Rugby World Cup?" The results are summarised:

Country	Number of supporters who think their country will win
Australia	116
England	13
France	25
New Zealand	140
Western Samoa	50
South Africa	47
Wales	24
Undecided	65
Total	480

Calculate a 90% confidence Interval for the proportion who had not yet decided.

$$\text{Solution: } p(\text{not decided}) = \frac{65}{480}; \quad 90\% \text{ confidence} \Rightarrow Z = 1.645$$

$$\Rightarrow C.I. = \frac{65}{480} \pm 1.645 \times \sqrt{\left(\frac{\left(\frac{65}{480}\right) \times \left(\frac{415}{480}\right)}{480}\right)} = 0.110, 0.161$$

We can be 90% confident between 11% and 16% of the population were undecided. (If you change the fractions to decimals, there will be a slight rounding error, but this will usually not be greatly significant.)

### Practice Questions

1. Samples of size  $n$  are taken from populations with a probability of success  $p$ . Use the values of  $n$  and  $p$ , the sample size and proportion, given below, to find confidence intervals for the population proportion with the levels of confidence indicated.

	$n$	$p$	Confidence Level
a	60	0.54	99%
b	200	0.4	90%
c	1000	0.45	86%

2. A political candidate finds that in a random sample of 300 constituents, 34% support her party. Find the 95% confidence interval for the support she in fact has.
3. Houses on a street are numbered from 1 to 627. Roimata takes a random sample of 40 houses. She finds that in 25 of them, there are more than 3 residents. Find a 90% confidence interval for the proportion of all houses in the street having more than three residents.
4. A toy manufacturer wants to test for the proportion of faulty toys in a large batch produced by a particular factory. He tests a random sample of 200 toys and finds that 25 are faulty. Calculate a 94% confidence interval for the proportion of faulty toys in the complete batch.
5. In a survey carried out in Auckland, 38 people out of a random sample of 70 people said that they bought the New Zealand Herald regularly. Find a 99% confidence interval for the proportion of people who buy the Herald in Auckland.

### Answers C.I.(mean)

1. Population variance given, so use  $Z_{\text{score}}$ , and calculate standard dev.  
 $\sigma^2 = 6.25 \Rightarrow \sigma = 2.5 \quad 99\% \Rightarrow Z = 2.58$   
*Hence*  $C.I. = 75.2 \pm 2.58 \times \frac{2.5}{\sqrt{20}} = (73.75, 76.64).$
2. Sample standard deviation given so use  $t_{\text{score}}$   
 $C.I. = 105 \pm 2.009 \times \frac{8}{\sqrt{60}} = (102.9, 107.1)$
3. Population std.dev. given, so use  $Z_{\text{score}}$ .
  - (a)  $90\% \Rightarrow Z = 1.645 \quad C.I. = 82 \pm 1.645 \times \frac{3.6}{\sqrt{100}} = (81.4, 82.6)$
  - (b)  $95\% \Rightarrow Z = 1.96 \quad C.I. = 82 \pm 1.96 \times \frac{3.6}{\sqrt{100}} = (81.3, 82.7)$
  - (c)  $99\% \Rightarrow Z = 2.58 \quad C.I. = 82 \pm 2.58 \times \frac{3.6}{\sqrt{100}} = (81.07, 82.93)$
4. Calculate sample mean and s.d.:  $\bar{x} = 15.17, s = 4.17$   
 $95\% \text{ and } df = 13 \Rightarrow t = 2.16 \quad C.I. = 15.17 \pm 2.16 \times \frac{4.17}{\sqrt{14}} = (12.76, 17.58)$
5. Calculate sample mean and s.d.:  $\bar{x} = 5.75, s = 1.768$   
 $80\% \text{ and } df = 19 \Rightarrow t = 1.328 \quad C.I. = 5.75 \pm 1.328 \times \frac{1.768}{\sqrt{20}} = (5.22, 6.28)$
6.  $\bar{x} = 50.5, s = \sqrt{400} = 20, n = 15 \Rightarrow df = 14, p = 0.025 \Rightarrow t = 2.1448$   
 $C.I. = 50.5 \pm 2.1448 \times \frac{20}{\sqrt{15}} = (39.42, 61.58)$

That is, we can be 95% confident, credit card users spent on average between \$40 and \$62.

$$7. \quad \bar{x} = 4.85, \quad s = 1.47$$

$$n = 20 \Rightarrow d.f. = 19, \quad p = 0.025 \Rightarrow t = 2.093$$

$$C.I. = 4.85 \pm 2.093 \times \frac{1.47}{\sqrt{20}} = (4.16, 5.54)$$

That is, we can be 95% confident that the true acceleration time is between 4.16 and 5.54 seconds.

$$8. \quad \bar{x} = 11.86, s = 0.1614 \text{ (after entering data in calculator).}$$

$$n = 7 \Rightarrow df = 6, p = 0.005 \Rightarrow t = 3.7074$$

$$C.I. = 11.86 \pm 3.7074 \times \frac{0.1614}{\sqrt{7}} = (11.63, 12.09)$$

That is we can be 99% confident that the true voltage in the power packs is between 11.63 and 12.09 volts.

### Answers *C.I.(proportion)*

$$1. \quad a \quad C.I. = 0.54 \pm 2.58 \times \sqrt{\left(\frac{0.54 \times 0.46}{60}\right)} = (0.37, 0.71)$$

$$b. \quad C.I. = 0.4 \pm 1.645 \times \sqrt{\frac{0.4 \times 0.6}{200}} = (0.34, 0.46)$$

$$c. \quad C.I. = 0.45 \pm 1.48 \times \sqrt{\frac{0.45 \times 0.55}{1000}} = (0.43, 0.47)$$

$$2. \quad n = 300, \quad p = 0.34, \quad 95\% \Rightarrow Z = 1.96$$

$$C.I. = 0.34 \pm 1.96 \times \sqrt{\frac{0.34 \times 0.66}{300}} = (0.29, 0.39)$$

$$3 \quad n = 40, \quad p = \frac{25}{40} = 0.625 \quad 90\% \Rightarrow Z = 1.645$$

$$C.I. = 0.625 \pm 1.645 \times \sqrt{\frac{0.625 \times 0.375}{40}} = (0.499, 0.751)$$

$$4. \quad C.I. = 0.125 \pm 1.88 \times \sqrt{\frac{0.125 \times 0.875}{200}} = (0.081, 0.169)$$

$$5. \quad C.I. = \frac{38}{70} \pm 2.58 \times \sqrt{\frac{\frac{38}{10} \times \frac{32}{70}}{70}} = (0.389, 0.696)$$



# To find a z-score

(Using Z table with upper half shading only)

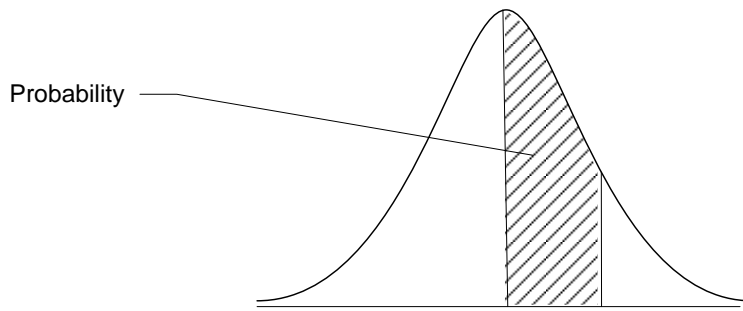


Table entry is probability  
at or below z

Z	.00	.01	.02	.03	.04	0.05	0.06	0.07	0.08	0.09
0.0										
0.1										
0.2										
....										
...										
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
etc										

To locate the appropriate Z-score

For example, to find the correct Z-score for a 98% confidence interval:

—	Change the %age to a decimal	98% = 0.98
—	Halve this decimal	$0.98/2 = 0.49$
—	Find this value in the table	Closest value = .4901
—	Use the corresponding Z-score	Z-score = 2.33

# To find a z-score

(Using Z table with left hand tail shaded)

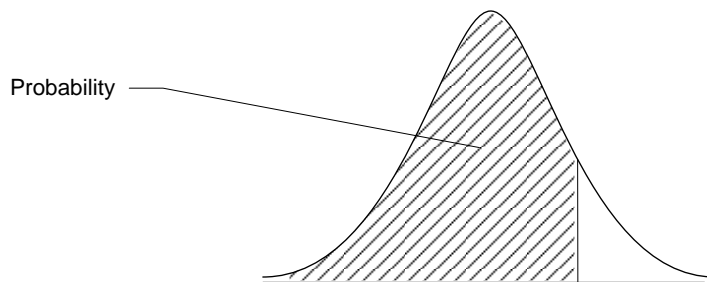


Table entry is probability  
at or below z

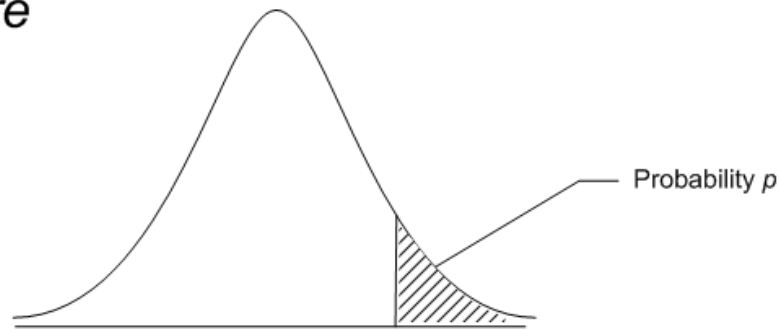
Z	.00	.01	.02	.03	.04	0.05	0.06	0.07	0.08	0.09
0.0										
0.1										
0.2										
....										
...										
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
etc										

To locate the appropriate Z-score

For example, to find the correct Z-score for a 98% confidence interval:

—	Change the %age to a decimal	98% = 0.98
—	Halve this decimal	$0.98/2 = 0.49$
—	Add 0.5	$0.49 + 0.5 = 0.99$
—	Find this value in the table	Closest value = .9901
—	Use the corresponding Z-score	Z-score = 2.33

## To find a *t*-score



df	.25	.15	.10	.05	.025	.02	0.01	0.005	0.001	0.005
1										
2										
3										
.....										
...										
19	.688	.1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.579	3.883
etc										

To locate the appropriate *t*-score

- Subtract the %age from 100%
- Change the %age to a decimal
- $p = \text{half this decimal}$
- Decide *df*
- Locate the intersection of *df* row and *p* column
- Use the corresponding *t*-score

For example, to find the correct *t*-score for a 98% confidence interval, with sample size = 20:

$$100\% - 98\% = 2\%$$

$$2\% = 0.02$$

$$p = 0.02/2 = 0.01$$

$$df = 19$$

$$t\text{-score} = 2.539$$